Basic tools and results: Value and equilibria

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Abstract

This lecture introduces strategic games and the main concepts: value and equilibria.
We provide several proofs of the minmax theorem based on dynamical processes:
- unilateral procedure in discrete time,
- continuous time and ODE,
- fictitious play in discrete and continuous time,
- replicator dynamics,
and the relation with approachability theory.
We then state existence proofs for equilibria and describe the links between equilibria and variational inequalities.
We introduce supermodular, potential and dissipative games.
Finally we present correlated equilibrium and give an elementary proof of existence.
An appendix is devoted to a short presentation of "stochastic approximation tools".

Strategic games: introduction

Strategic games: notations

A strategic game G is defined by:
- a set I of players, (I will denote also the cardinal)
- a set S^i of strategies for each player $i \in I$
- a mapping g from $S = \Pi_{i=1}^{I} S^i$ into $I\!R^I$.

$g^i(s)$ is the payoff of player i when the profile $s = (s^1, \cdots, s^I)$ is played.

Denote $s = (s^i, s^{-i})$ where s^{-i} is the vector $\{s^j\}$ for $j \neq i$ and $S^{-i} = \Pi_{j \neq i} S^j$.

$\Delta(K)$= simplex on a finite set K = $\{x \in I\!R^K, x^k \geq 0, \sum_{k \in K} x^k = 1\}$, set of Borel probabilities on a topological space K (compact, metric).

Mixed extension of G:
$\Sigma^i = \Delta(S^i)$, $i \in I$ set of mixed strategies of player i, $\Sigma = \Pi_{i \in I} \Sigma^i$, multilinear extension of g to Σ (assuming Fubini):

$$g^j(\sigma) = \int_{\Pi_{i=1}^{I} S^i} g^j(s) \Pi_{i=1}^{I} d\sigma^i(s^i)$$

Strategic games: definitions

For $\varepsilon \geq 0$, the $(\varepsilon-)$ best reply correspondence BR^i_ε of player i, from S^{-i} to S^i, is defined by:

$$BR^i_\varepsilon(s^{-i}) = \{s^i \in S^i : g^i(s^i, s^{-i}) \geq g^i(t^i, s^{-i}) - \varepsilon, \forall t^i \in S^i\}.$$

It associates to every profile of the opponents the set of $\varepsilon-$best replies of a player.
Write $BR : S \rightrightarrows S$, for the global best reply correspondence that maps $s \in S$ to $\prod_{i \in I} BR^i(s^{-i})$.
The extension of $BR : \Sigma \rightrightarrows \Sigma$ to the mixed game is straightforward.

Equilibrium

A Nash equilibrium (Nash, 1950) is a profile of strategies $s \in S$ where no player can gain by changing his strategy.
More generally, for $\varepsilon \geq 0$, an $\varepsilon-$equilibrium is a profile $s \in S$, such that for all i, $s^i \in BR^i_\varepsilon(s^{-i})$, which is:

$$g^i(t^i, s^{-i}) \leq g^i(s) + \varepsilon, \quad \forall t^i \in S^i, \quad \forall i.$$

Thus s is a Nash equilibrium iff s is a fixed point of the BR correspondence:

$$s \in BR(s).$$

An equilibrium s is strict if $\{s\} = BR(s)$.
Alternatively, a profile t eliminates a profile s is there exists a player $i \in I$ with $g^i(t^i, s^{-i}) > g^i(s)$. Let $E(t) \subset S$ be the set of profiles not eliminated by t.
An equilibrium is then a profile in $\cap_{t \in S} E(t)$.
This formulation is in the spirit of an equilibrium being a "rational" rule of behavior.

Zero-sum games: value

A two-person, zero-sum game is a game where $I = 2$, $S^1 = S$, $S^2 = T$ and given $f : S \times T \to I\!R$, the payoffs are $g^1 = -g^2 = f$.
The interests of the players are opposite: $g^1 + g^2 = 0$.
One introduces the following quantities:

$$\underline{v} = \sup_S \inf_T f(s,t) \qquad \overline{v} = \inf_T \sup_S f(s,t);$$

and a strategy s is $\varepsilon(\geq 0)$-optimal if:

$$f(s,t) \geq \underline{v} - \varepsilon, \quad \forall t \in T.$$

The game has a value v if: $\underline{v} = \overline{v} = v$.

The link between value and equilibria is as follows:

Proposition

Assume that the game has a value and that s,t are ε-optimal. Then they form a $2\varepsilon-$equilibrium:

$$f(s,t') + 2\varepsilon \geq f(s,t) \geq f(s',t) - 2\varepsilon, \quad \forall s',t' \in S \times T.$$

(For $\varepsilon = 0$, this is a saddle point.)

Minmax theorem 1

Finite case

The sets of actions $S = I$, $T = J$ are finite. The (payoff of the) game G is represented by a $I \times J$ matrix A, an element $x \in \Delta(I)$ corresponds to a row matrix (mixed strategy of player 1) and an element $y \in \Delta(J)$ to a column matrix (mixed strategy of player 2), so that the payoff is the bilinear form $f(x,y) = xAy$.

Theorem (Von Neumann, 1928)

Let A be a $I \times J$ real matrix.
There exist (x^*, y^*, v) in $\Delta(I) \times \Delta(J) \times \mathbb{R}$ such that :

$$x^*Ay \geq v, \qquad \forall y \in \Delta(J) \qquad \text{and} \qquad xAy^* \leq v, \ \forall x \in \Delta(I). \qquad (1)$$

In other words, the mixed extension of a matrix game has a value (one also says that any finite zero-sum game has a value in mixed strategies) and both players have optimal strategies.

The real number v in the theorem is uniquely determined and corresponds to the value of the matrix A :

$$v = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} xAy = \min_{y \in \Delta(J)} \max_{x \in \Delta(I)} xAy.$$

It is also denoted by $\mathtt{val}(A)$.
As a mapping defined on matrices, from $I\!R^{I \times J}$ to $I\!R$, the operator $\mathtt{val}$ is continuous, monotonic (increasing) and non expensive :

$$|\mathtt{val}(A) - \mathtt{val}(B)| \leq \|A - B\|_\infty$$

These properties extend to the general framework of zero-sum games: $|\mathtt{val}(f) - \mathtt{val}(g)| \leq \|f - g\|_\infty$.

Minmax theorem 2

Theorem (Sion, 1958)

Let $G = (S,T,f)$ be a zero-sum game satisfying:
(i) S and T are convex,
(ii) S or T is compact,
(iii) for each t in T, $f(.,t)$ is quasi-concave u.s.c. in s, and for each s in S, $f(s,.)$ is quasi-convex l.s.c. in t.
Then G has a value: $\sup_{s\in S}\inf_{t\in T} f(s,t) = \inf_{t\in T}\sup_{s\in S} f(s,t)$.

Theorem (Mixed extension)

Let $G = (S,T,f)$ be a zero-sum game such that:
(i) S and T are compact Hausdorff topological spaces,
(ii) for each t in T, $f(.,t)$ is u.s.c., and for each s in S, $f(s,.)$ is l.s.c.
(iii) f is bounded and mesurable with respect to the product Borel σ-algebra $\mathscr{B}_S \otimes \mathscr{B}_T$.
Then the mixed extension $(\Delta(S),\Delta(T),f)$ of G has a value. Each player has a mixed optimal strategy, and for each $\varepsilon > 0$ each player has an ε-optimal strategy with finite support.

Minmax theorem 2

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Then G *has a value:* $\sup_{s\in S}\inf_{t\in T}f(s,t)=\inf_{t\in T}\sup_{s\in S}f(s,t)$.

Theorem (Mixed extension)

Let $G=(S,T,f)$ *be a zero-sum game such that:*
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Minmax principle

The next example, due to Aumann and Maschler, 1968 [3] shows the difference between an analysis in terms of maxmin/minmax optimal strategies or of equilibria.

	L	R
T	$(2,0)$	$(0,1)$
B	$(0,1)$	$(1,0)$

Considering the payoff of player 1, one faces a zero-sum game with value $V_1 = 2/3$ and maxim optimal strategy for player 1: $\bar{x} = (1/3, 2/3)$.

The dual parameters are $V_2 = 1/2$ and $\bar{y} = (1/2, 1/2)$ for player 2.

On the other hand the game has a single equilibrium $x^* = (1/2, 1/2), y^* = (1/3, 2/3)$ with payoff $E = (2/3, 1/2)$.

Note that for player 1 the equilibrium payoff is equal to his value (2/3) but that the equilibrium strategy x^* does not guarantee it, while $\bar{x}$ does. A similar statement holds for player 2.

However the strategies $(\bar{x}, \bar{y})$ are not in equilibrium.

The next 4 sections give proofs of the minmax theorem (finite case)

Unilateral process

A) *Preliminary result*

Le C be a non empty closed convex subset of $I\!R^k$ (endowed with the Euclidean norm) and $\{z_n\}$ a bounded sequence in $I\!R^k$. For $z \in I\!R^k$, $\Pi_C(z)$ stands for the closest point to z in C (which is also the projection of z on C) and $\bar{z}_n$ is the Cesaro mean up to stage n of the sequence $\{z_m\}$:

$$\bar{z}_n = \frac{1}{n} \sum_{m=1}^{n} z_m.$$

$\{z_n\}$ is a Blackwell C-sequence (Blackwell, 1956 [8]) if it satisfies :

$$\langle z_{n+1} - \Pi_C(\bar{z}_n), \bar{z}_n - \Pi_C(\bar{z}_n)\rangle \leq 0, \quad \forall n. \tag{2}$$

Claim: $d_n = d(\bar{z}_n, C)$ converges to 0.

Proof: 1) Let $u_n = \Pi_C(\bar{z}_n)$then :

$$d_{n+1}^2 \leq \|\bar{z}_{n+1} - u_n\|^2 = \|\bar{z}_n - u_n\|^2 + \|\bar{z}_{n+1} - \bar{z}_n\|^2 + 2\langle \bar{z}_{n+1} - \bar{z}_n, \bar{z}_n - u_n\rangle$$

Decompose:

$$\langle \bar{z}_{n+1} - \bar{z}_n, \bar{z}_n - u_n \rangle = (\frac{1}{n+1}) \langle z_{n+1} - \bar{z}_n, \bar{z}_n - u_n \rangle$$

$$= (\frac{1}{n+1})(\langle z_{n+1} - u_n, \bar{z}_n - u_n \rangle - \|\bar{z}_n - u_n\|^2).$$

Using the hypothesis we obtain:

$$d_{n+1}^2 \leq (1 - \frac{2}{n+1})\, d_n^2 + (\frac{1}{n+1})^2 \|z_{n+1} - \bar{z}_n\|^2.$$

From:

$$\|z_{n+1} - \bar{z}_n\|^2 \leq 2\|z_{n+1}\|^2 + 2\|\bar{z}_n\|^2 \leq 4M^2,$$

one deduces:

$$d_{n+1}^2 \leq (\frac{n-1}{n+1})\, d_n^2 + (\frac{1}{n+1})^2\, 4M^2$$

and by induction :

$$d_n \leq \frac{2M}{\sqrt{n}}.$$

B) *Consequence: minmax's theorem*.

Let A be a $I \times J$ matrix and assume that the minmax is 0 :

$$\bar{v} = \min_{y \in \Delta(J)} \max_{i \in I} e^i A y = 0.$$

Claim : Player 1 can guarantee 0, i.e. $\underline{v} \geq 0$.

Proof : Construct by induction a sequence $z_n \in \mathbb{R}^J$.

The first term z_1 is any row of the matriz A. Given $z_1, z_2, ..., z_n$, define z_{n+1} as follows : Let $\bar{z}_n^+$ be the vector with j^{th} coordinate equals to $\max(\bar{z}_n^j, 0)$.

If $\bar{z}_n = \bar{z}_n^+$, take z_{n+1} as any row of A.

Otherwise let $a > 0$ such that :

$$y_n = \frac{\bar{z}_n^+ - \bar{z}_n}{a} \in \Delta(J).$$

Since $\bar{v} = 0$, there exists $i_{n+1} \in I$ such that $e^{i_{n+1}} A y_n \geq 0$. Define z_{n+1} as the line i_{n+1} of the matrix A.

By construction: $0 \leq ae^{i_{n+1}}Ay_n = \langle z_{n+1}, \bar{z}_n^+ - \bar{z}_n \rangle$.
Since $\langle \bar{z}_n^+, \bar{z}_n^+ - \bar{z}_n \rangle = 0$ one gets :

$$\langle z_{n+1} - \bar{z}_n^+, \bar{z}_n - \bar{z}_n^+ \rangle \leq 0. \tag{3}$$

Let $C = \{z \in I\!R^k;\ \ z \geq 0\}$ and note that : $\bar{z}_n^+ = \Pi_C(\bar{z}_n)$ so that (3) gives (2): $\{z_n\}$ is a Blackwell C-sequence.
Finally write $\bar{z}_n = \bar{x}_n A$. Any accumulation point $\bar{x}$ of the sequence $\{\bar{x}_n\} \in \Delta(I)$ satisfies $\bar{x}A \in C$ hence $\bar{x}Ay \geq 0$, $\forall y \in \Delta(J)$ and $\underline{v} \geq 0$. ■

Minmax theorem via ODE

We follow Brown and von Neumann, 1950 [10]

A) Claim : Any real matrix B, $I \times I$ and antisymmetric ($B = -{}^tB$) has a value.

Proof : This is equivalent to find $x \in X(B) = \{x \in X \text{ with } Bx \leq 0\}$.

Let $K^i(x) = [e^iBx]^+, \bar{K}(x) = \sum_i K^i(x)$ and consider the dynamical system on X defined by:

$$\dot{x}_t^i = K^i(x_t) - x_t^i \bar{K}(x_t) \qquad (*)$$

Let $V(x) = \sum_i K^i(x)^2$. The set of rest points of $(*)$ is:

$X(B) = V^{-1}(0)$ since $K^i(x) = x^i \bar{K}(x)$ gives $V(x) = \bar{K}(x)xBx = 0$.

Finally:

$$\frac{d}{dt}V(x_t) = 2\sum_i K^i(x_t)e^iB\dot{x}_t = 2[K(x_t)BK(x_t) - \{K(x_t)Bx_t\}\bar{K}(x_t)] = -2\bar{K}(x_t)V(x_t).$$

Hence $V(x_t)$ is strictly decreasing on the complement of $X(B)$.

Compactness implies that the accumulation points of x_t are in $X(B)$ which is thus non empty. ∎

B) We now deduce from A) that any matrix A has a value.
One can assume $A_{ij} > 0$ for all (i,j).
B.a) Following (Gale, Kuhn and Tucker, 1950 [13]), introduce the antisymmetric matrix B, of size $(I+J+1) \times (I+J+1)$ defined by:

$$B = \begin{pmatrix} 0 & A & -1 \\ -{}^tA & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Consider an optimal strategy $z = (x, y, t)$ for player 1 in the game B, then x and y (normalized) are optimal strategies for both players in the game A.
B.b) An alternative proof (Brown and von Neumann) is to consider the $(I \times J) \times (I \times J)$ matrix C defined by:

$$C_{ij;i'j'} = A_{ij'} - A_{i'j}.$$

Hence each player plays in game C both as player 1 and 2 in the initial game A.
From an optimal strategy in game C one constructs optimal strategies for both players in game A.

Fictitious play

Let A be a $I \times J$ real matrix.
The following process, called fictitious play, has been introduced by Brown (1951) [9].

Consider two players playing in a repeated way the matrix game A. At each stage, each player is aware of the previous action of his opponent and compute the empirical distribution of the actions used in the past. He plays then an action which is a best reply to this average.

Explicitly, starting with any (i_1, j_1) in $I \times J$, consider at each stage n, $x_n = \frac{1}{n}\sum_{t=1}^{n} i_t$, viewed as an element of $\Delta(I)$, and similarly $y_n = \frac{1}{n}\sum_{t=1}^{n} j_t \in \Delta(J)$.

Definition

A sequence $(i_n, j_n)_{n \geq 1}$ with values in $I \times J$ is the realization of a fictitious play process for the matrix A if, for each $n \geq 1$, i_{n+1} is a best reply of player 1 to y_n for A:

$$i_{n+1} \in BR^1(y_n) = \{i \in I : e^i A y_n \geq e^k A y_n, \forall k \in I\}$$

and j_{n+1} is a best reply of player 2 to x_n for A ($j_{n+1} \in BR^2(x_n)$, defined in a dual way).

The main properties of this procedure are given by the next result.

Theorem

[Robinson, 1951, [37]] Let $(i_n, j_n)_{n\geq 1}$ be the realization of a fictitious play process for the matrix A.
1) The distance from (x_n, y_n) to the set of optimal strategies $X(A)\times Y(A)$ goes to 0, as $n\to\infty$.
In other words: $\forall \varepsilon>0, \exists N, \forall n\geq N, \forall x\in\Delta(I), \forall y\in\Delta(J)$:

$$x_n A y \geq \mathtt{val}(A)-\varepsilon \ \text{ and } \ x A y_n \leq \mathtt{val}(A)+\varepsilon.$$

2) The average payoff on the trajectory, namely $\frac{1}{n}\sum_{t=1}^{n} A_{i_t, j_t}$, converges to $\mathtt{val}(A)$.

We will prove the theorem by considering the continuous time analog.

Take as variables the empirical frequencies x_n and y_n, so that the discrete dynamics for player 1 writes :

$$x_{n+1} = \frac{1}{n+1}[i_{n+1} + nx_n] \qquad \text{with} \qquad i_{n+1} \in BR^1(y_n)$$

hence satisfies :

$$x_{n+1} - x_n \in \frac{1}{n+1}[BR^1(y_n) - x_n].$$

The corresponding system in continuous time is now :

$$\dot{x}_t \in \frac{1}{t}\left[BR^1(y_t) - x_t\right].$$

This is a differential inclusion which defines, with a similar condition for player 2, the process CFP: continuous fictitious play.

Write the payoff as $f(x,y) = xAy$ and for $(x,y) \in \Delta(I) \times \Delta(J)$, let :

$$L(y) = \max_{x' \in \Delta(I)} f(x',y) \qquad M(x) = \min_{y' \in \Delta(J)} f(x,y').$$

The duality gap at (x,y) is defined as : $W(x,y) = L(y) - M(x) \geq 0$ and the pair (x,y) defines optimal strategies in A if and only if $W(x,y) = 0$.

Proposition (Harris, 1998 [18]; Hofbauer and Sorin, 2006 [24])

For the CFP process, the duality gap converges to 0 at a speed $O(1/t)$.

Proof: Make the time change $z_t = x_{e^t}$ which leads to the autonomous differential inclusion:

$$\dot{x}_t \in \left[BR^1(y_t) - x_t\right], \quad \dot{y}_t \in \left[BR^2(x_t) - y_t\right].$$

known as the best reply dynamics (Gilboa and Matsui, 1991) [14].

Let now $(x_t, y_t)_{t\geq 0}$ be a solution of CFP. Denote by $w_t = W(x_t, y_t)$ the evaluation of the duality gap on the trajectory, and write $\alpha_t = x_t + \dot{x}_t \in BR^1(y_t)$ and $\beta_t = y_t + \dot{y}_t \in BR^2(x_t)$.
One has $L(y_t) = f(\alpha_t, y_t)$, thus:

$$\frac{d}{dt}L(y_t) = \dot{\alpha}_t D_1 f(\alpha_t, y_t) + \dot{y}_t D_2 f(\alpha_t, y_t).$$

The enveloppe's theorem (see e.g., Mas Colell, Whinston and Green, 1995, p. 964 [31]) shows that the first term collapses and the second term is $f(\alpha_t, \dot{y}_t)$ (since f is linear w.r.t. the second variable). Then we obtain :

$$\begin{aligned}
\dot{w}(t) &= \frac{d}{dt}L(y_t) - \frac{d}{dt}M(x_t) = f(\alpha_t, \dot{y}_t) - f(\dot{x}_t, \beta_t) \\
&= f(x_t, \dot{y}_t) - f(\dot{x}_t, y_t) = f(x_t, \beta_t) - f(\alpha_t, y_t) \\
&= M(x_t) - L(y_t) = -w(t)
\end{aligned}$$

thus : $w_t = w(0)\, e^{-t}$. There is convergence of w_t to 0 at exponential speed, hence convergence to 0 at a speed $O(1/t)$ in the original problem before the time change.

The convergence to 0 of the duality gap implies by uniform continuity the convergence of (x_t, y_t) to the set of optimal strategies $X(A) \times Y(A)$. ∎

Let us remark that by compactness of the sets of mixed strategies, one obtains the existence of optimal strategies in the matrix game (accumulation points of the trajectories). This provides an alternative proof of the minmax theorem, starting from the existence of a solution to CFP.

The result is actually stronger: the set $X(A) \times Y(A)$ is a global attractor for the best reply dynamics, which implies the convergence of the discrete time version, hence of the fictitious play process (Hofbauer and Sorin, 2006 [24]), i.e. part $1)$ of the Theorem.

We finally prove part $2)$ of the theorem.

Proof: Let us consider the sum of the realized payoffs : $R_n = \sum_{p=1}^{n} f(i_p, j_p)$. Writing : $U_m^i = \sum_{k=1}^{m} f(i, j_k)$, one obtains :

$$R_n = \sum_{p=1}^{n} (U_p^{i_p} - U_{p-1}^{i_p}) = \sum_{p=1}^{n} U_p^{i_p} - \sum_{p=1}^{n-1} U_p^{i_{p+1}} = U_n^{i_n} + \sum_{p=1}^{n-1} (U_p^{i_p} - U_p^{i_{p+1}})$$

but the fictitious play property implies that :

$$U_p^{i_p} - U_p^{i_{p+1}} \leq 0.$$

Hence $\limsup_{n\to\infty} \frac{R_n}{n} \leq \limsup_{n\to\infty} \max_i \frac{U_n^i}{n} \leq \mathtt{val}(A)$, since $\frac{U_n^i}{n} = f(i, y_n) \leq \mathtt{val}(A) + \varepsilon$ for n large enough by part 1) of Theorem 5.

The dual inequality thus implies the result. ∎

Note that Part 1 and Part 2 of the theorem are independent. In general convergence of the average marginal trajectories on moves does not imply property of the average payoff on the trajectory (for example in matching pennies convergence in average strategies to $(1/2, 1/2)$ is compatible with a sequence of payoffs 1 or -1).

Replicator dynamics

Consider the replicator dynamics defined by the following equations, with $x_0 \in int(X), y_0 \in int(Y)$:

$$\dot{x}_t^i = x_t^i[e^i A y_t - x_t A y_t], \qquad \forall i \in I \tag{4}$$

$$\dot{y}_t^j = y_t^j[-x_t A e^j + x_t A y_t], \qquad \forall j \in J.$$

This defines trajectories in $X \times Y$ since $\frac{d}{dt}\sum_i x_t^i = 0$ and $x_0^i > 0$ implies $x_t^i > 0, \forall t \geq 0$.
Introduce $\bar{x}_T = \frac{1}{T}\int_0^T x_t dt$.
By integrating one obtains:

$$\frac{1}{T}[\log x_T^i - \log x_0^i] = e^i A \bar{y}_T - \frac{1}{T}\int_0^T x_s A y_s ds \qquad \forall i \in I.$$

Consider a sequence $T_k \to \infty$ on which $(\bar{x}_{T_k}, \bar{y}_{T_k}, \frac{1}{T_k}\int_0^{T_k} x_s A y_s ds)$ converge to (x^*, y^*, w). Then:

$$e^i A y^* \leq w \leq x^* A e^j, \qquad \forall i \in I, \qquad \forall j \in J.$$

Hence the game has a value, w and x^*, y^* are optimal strategies.
The proof shows more:
any accumulation point $\bar{x}$ of $\bar{x}_T$ belongs to $X(A)$,
$\frac{1}{T}\int_0^T x_s A y_s ds$ converges to the value, Hofbauer, 2018 [22].

Approachability theory

Let A be a $I \times J$-matrix with entries in $I\!R^K$: $A_{ij} \in I\!R^k$ is the vector payoff, or outcome, if player 1 plays i and player 2 plays j.
Given $s \in \Delta(I)$, we denote by sA the subset of $I\!R^K$ of feasible expected vector payoffs when player 1 plays the mixed action s:

$$\begin{aligned} sA &= \{z \in I\!R^k : \exists t \in \Delta(J) \quad \text{s.t.} \quad z = sAt\} \\ &= \{\sum_{i \in I, j \in J} s^i A_{ij} t^j, t \in \Delta(J)\} = Conv\{\sum_{i \in I} s^i A_{ij}, j \in J\} \end{aligned}$$

Let C be a closed convex subset of $I\!R^k$.
Assume that C is a **B**-set for player 1, i.e. satisfies:

$$\forall z \notin C, \exists s \in \Delta(I) \text{ s.t. } \forall u \in sA : \quad \langle u - \Pi_C(z), z - \Pi_C(z) \rangle \leq 0. \quad (5)$$

Geometrically, the affine hyperplane containing $\Pi_C(z)$ and orthogonal to $[z, \Pi_C(z)]$ separates z from sA.

The game is played in discrete time for infinitely many stages: at each stage $n = 1, 2, ...$, after having observed the past history h_{n-1} of actions chosen from stage 1 to stage $n-1$, i.e. $h_{n-1} = (i_1, j_1, ..., i_{n-1}, j_{n-1}) \in \mathcal{H}_{n-1} = (I \times J)^{n-1}$, player 1 chooses $s_n(h_{n-1}) \in \Delta(I)$ and player 2 chooses $t_n(h_{n-1}) \in \Delta(J)$. Then a couple $(i_n, j_n) \in I \times J$ is selected according to the product probability $s_n(h_{n-1}) \otimes t_n(h_{n-1})$, and the game goes to stage $n+1$ with the history $h_n = (i_1, j_1, ..., i_n, j_n) \in \mathcal{H}_n$.

A strategy σ of player 1 in the repeated game is a sequence $\sigma = (s_1, ..., s_n, ...)$ with $s_n : \mathcal{H}_{n-1} \to \Delta(I)$ for each n. (Similarly for 2). A couple (σ, τ) naturally defines a probability distribution $\mathbf{P}_{\sigma,\tau}$ over the set of plays $\mathcal{H}_\infty = (I \times J)^\infty$, endowed with the product σ-algebra, and $\mathbf{E}_{\sigma,\tau}$ is the associated expectation. Every play $h = (i_1, j_1, ..., i_n, j_n, ...)$ of the game induces a sequence of vector payoffs $z(h) = (z_1 = A_{i_1 j_1}, ..., z_n = A_{i_n j_n}, ...)$ with values in $I\!R^k$.

We denote by $\bar{z}_n$ the Cesaro-average payoff up to stage n:

$$\bar{z}_n(h) = \frac{1}{n}\sum_{k=1}^{n} A_{i_k j_k} = \frac{1}{n}\sum_{k=1}^{n} z_k.$$

Blackwell (1956) [8] constructed a strategy σ of player 1 which generates a probability on plays $h = (i_1, j_1, \ldots, i_n, j_n, \ldots)$ such that:
Claim:
$\bar{z}_n(h)$ converges to C, whatever the strategy τ of player 2:

$$d_n = \|\bar{z}_n - \Pi_C(\bar{z}_n)\| \xrightarrow[n\to\infty]{} 0, \mathbf{P}_{\sigma,\tau} - \text{a.s.}.$$

Blackwell's strategy σ is defined inductively as follows, using the fact that C is a **B**-set. At stage $n+1$, play $s_{n+1} \in \Delta(I)$ such that for each $t \in \Delta(J)$:

$$\langle s_{n+1}At - \Pi_C(\bar{z}_n), \bar{z}_n - \Pi_C(\bar{z}_n)\rangle \leq 0.$$

Notice that if $\bar{z}_n \in C$, any s_{n+1} will do and player 1 can play arbitrarily.

Write $\mathbf{E}$ for $\mathbf{E}_{\sigma,\tau}$. One has :

$$\begin{aligned} d_{n+1}^2 &\leq \|\bar{z}_{n+1} - \Pi_C(\bar{z}_n)\|^2, \\ &\leq \|\frac{1}{n+1}(z_{n+1} - \Pi_C(\bar{z}_n)) + \frac{n}{n+1}(\bar{z}_n - \Pi_C(\bar{z}_n))\|^2, \\ &\leq \left(\frac{1}{n+1}\right)^2 \|z_{n+1} - \Pi_C(\bar{z}_n)\|^2 + \left(\frac{n}{n+1}\right)^2 {d_n}^2 \\ &\quad + \frac{2n}{(n+1)^2} < z_{n+1} - \Pi_C(\bar{z}_n), \bar{z}_n - \Pi_C(\bar{z}_n) > . \end{aligned}$$

By assumption, the conditional expectation of the above scalar product is non positive, so :

$$\mathbf{E}\left(d_{n+1}^2 | h_n\right) \leq \frac{1}{(n+1)^2} \mathbf{E}\left(\|z_{n+1} - \Pi_C(\bar{z}_n)\|^2 | h_n\right) + \left(\frac{n}{n+1}\right)^2 {d_n}^2.$$

$\mathbf{E}(\ \langle z_{n+1} - \Pi_C(\bar{z}_n), \bar{z}_n - \Pi_C(\bar{z}_n)\rangle\ |h_n) \leq 0$ implies:
$\mathbf{E}\left(\|z_{n+1} - \Pi_C(\bar{z}_n)\|^2 | h_n\right) \leq \mathbf{E}\left(\|z_{n+1} - \bar{z}_n\|^2 | h_n\right) \leq (2\|A\|_\infty)^2$
with $\|A\|_\infty = \max_{i,j,k} \left\| A_{ij}^k \right\|$.
Thus:

$$\mathbf{E}\left(d_{n+1}^2 | h_n\right) \leq \left(\frac{n}{n+1}\right)^2 {d_n}^2 + \left(\frac{1}{n+1}\right)^2 4\|A\|_\infty^2. \tag{6}$$

Taking expectation leads to: $\forall n \geq 1$,
$\mathbf{E}\left(d_{n+1}^2\right) \leq (\frac{n}{n+1})^2 \mathbf{E}({d_n}^2) + (\frac{1}{n+1})^2 4\|A\|_\infty^2$.
Then by induction, for each $n \geq 1$ we have $\mathbf{E}(d_n^2) \leq \frac{4\|A\|_\infty^2}{n}$, finally

$$\mathbf{E}(d_n) \leq \frac{2\|A\|_\infty}{\sqrt{n}}.$$

In particular, the convergence is uniform in τ.

Define $e_n = d_n^2 + \sum_{k=n+1}^{\infty} \frac{4\|A\|_\infty^2}{k^2}$ for each n. Inequality (6) yields :

$$\mathbf{E}(e_{n+1}|h_n) \le e_n,$$

thus $\{e_n\}$ is a non negative supermartingale with expectation converging to 0.
So $e_n \xrightarrow[n\to\infty]{} 0$ $\mathrm{P}_{\sigma,\tau}$ a.s., and finally $d_n \xrightarrow[n\to\infty]{} 0$ $\mathrm{P}_{\sigma,\tau}$ a.s. ∎

Convex case
C convex is approachable iff $\forall t \in \Delta(J)$:

$$At \cap C \neq \emptyset$$

Equilibrium: existence

The main result is the following:

Theorem (Nash, 1951 [36], Glicksberg, 1952 [15], Fan, 1952 [11])

1) If S^i is a compact convex subset of a topological vector space, g^i is continuous and quasi concave w.r.t. s^i, for all $i \in I$, the set of equilibria is compact and non empty.
2) If S^i is compact , g^i continuous, for all $i \in I$, the mixed extension of the game has an equilibrium.

Proof: 1) By continuity and compactness, for each profile t, the set $E(t)$ (of profiles not eliminated by t) is a compact subset of S. By the intersection property, to prove the existence of an equilibrium, it is enough to show that for any finite family $\{t(k), k \in K\}$, of profiles in S, the intersection $\cap_{k \in K} E(t(k))$ is not empty.
We are then in a finite dimensional framework (replace each S^i by $co(\{t^i(k), k \in K\})$) and an equilibrium of the reduced game will be in $\cap_{k \in K} E(t(k))$.

Now g quasi-concave implies that for all s, $BR(s)$ is convex. By continuity and compactness, $BR(s)$ is compact and non-empty for each s. The joint continuity hypothesis implies that the graph of the correspondence BR is closed.
Then use Ky Fan's fixed point theorem for the correspondence BR on S. The corresponding fixed point is an equilibrium.

2) If S^i is compact, $\Sigma^i = \Delta(S^i)$ is convex and compact (for the weak$*$ topology). Similarly if g^i is continuous on S, its extension to $\Sigma = \prod_{j \in I} \Sigma^j$ is continuous (again for the weak$*$ topology using for example the Stone-Weierstrass theorem to get the joint continuity) and multilinear. Then use part 1. ∎

Equilibrium : finite case

We consider here the case of a finite game: finitely many players, each player $i \in I$ having finitely many strategies in S^i. The finiteness assumption allows for a more precise analysis of equilibria.

Lemma

σ is a mixed equilibrium iff for all i and all $s^i \in S^i$:

$$g^i(s^i, \sigma^{-i}) < \max_{t^i \in S^i} g^i(t^i, \sigma^{-i}) \Rightarrow \sigma^i(s^i) = 0.$$

Theorem (Nash, 1950 [35])

Every finite game G has a mixed equilibrium.

Proof: Define the Nash map f from Σ to Σ by:

$$f(\sigma)^i(s^i) = \frac{\sigma^i(s^i) + (g^i(s^i, \sigma^{-i}) - g^i(\sigma))^+}{1 + \sum_{t^i \in S^i} (g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+}$$

where $a^+ = max(a, 0)$.

f is well defined and with values in Σ: $f(\sigma)^i(s^i) \geq 0$ and $\sum_{s^i \in S^i} f(\sigma)^i(s^i) = 1$.
Since f is continuous and Σ convex, compact, Brouwer's fixed point theorem implies the existence of $\sigma \in \Sigma$ with $f(\sigma) = \sigma$.
Let us prove that such σ is an equilibrium.
Otherwise there exists $i \in I$ and $u^i \in S^i$ with $g^i(u^i, \sigma^{-i}) - g^i(\sigma) > 0$ hence $\sum_{t^i \in S^i}(g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+ > 0$. Since there exists s^i with $\sigma^i(s^i) > 0$ and $g^i(s^i, \sigma^{-i}) \leq g^i(\sigma)$ one obtains:

$$\sigma^i(s^i) = f(\sigma)^i(s^i) = \frac{\sigma^i(s^i)}{1 + \sum_{t^i \in S^i}(g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+} < \sigma^i(s^i)$$

hence a contradiction.
Note that reciprocally any equilibrium is a fixed point of f since all quantities $(g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+$ vanish. ∎

Recall that each S^i is finite, with cardinal m^i. Let $m = \Pi_i m^i$.
A game can thus be identified with a point in $I\!R^{Nm}$. For example, with 2 players having each 2 strategies one obtains $g \in I\!R^8$ specified by:

	L	R
T	(a_1, a_2)	(a_3, a_4)
B	(a_5, a_6)	(a_7, a_8)

Proposition

The set of equilibria is defined by a finite family of large polynomial inequalities.

Proof: σ is an equilibrium iff:

$$\sum_{s^i \in S^i} \sigma^i(s^i) - 1 = 0, \quad \sigma^i(s^i) \geq 0, \quad \forall s^i \in S^i, \forall i \in I,$$

$$g^i(\sigma) = \sum_{s=(s^1,\dots,s^N) \in S} \prod_i \sigma^i(s^i) g^i(s) \geq g^i(t^i, \sigma^{-i}), \forall t^i \in S^i, \forall i \in I,$$

the unknown being the family $\{\sigma^i(s^i)\}$.
We used the linearity to make the comparison only to extreme points.

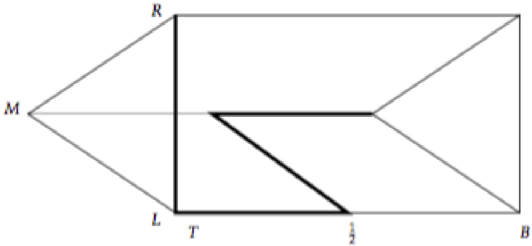
Corollary

The set of equilibria is semi-algebraic.
It is a finite union of closed connected components.

Example 1

	L	M	R
T	$(2,1)$	$(1,0)$	$(1,1)$
B	$(2,0)$	$(1,1)$	$(0,0)$

The set of equilibria is described by the thick line below.

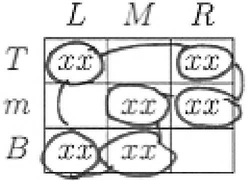


There is one connected component, homeomorphic to a segment.

Example 2 (Kohlberg and Mertens, 1986 [27])

	L	M	R
T	$(1,1)$	$(0,-1)$	$(-1,1)$
m	$(-1,0)$	$(0,0)$	$(-1,0)$
B	$(1,-1)$	$(0,-1)$	$(-2,-2)$

There is only one connected component of equilibria which is of the form:



hence homeomorphic to a circle in Σ.

In addition each point is limit of a sequence of equilibria of close-by games, like the next one, with $\varepsilon > 0$:

	L	M	R
T	$(1, 1-\varepsilon)$	$(\varepsilon, -1)$	$(-1-\varepsilon, 1)$
m	$(-1, -\varepsilon)$	$(-\varepsilon, \varepsilon)$	$(-1+\varepsilon, -\varepsilon)$
B	$(1-\varepsilon, -1)$	$(0, -1)$	$(-2, -2)$

with equilibrium $[(\varepsilon/(1+\varepsilon), 1/(1+\varepsilon), 0); (0, 1/2, 1/2)]$ close to $[(0, 1, 0); (0, 1/2, 1/2)]$.

Nash equilibria and variational inequalities

For various classes of games, Nash equilibria can be represented as solutions of variational inequalities.

Finite games

Define the vector payoff function $Vg^i : \Sigma^{-i} \longrightarrow I\!R^{S^i}$ by $Vg^i(\sigma^{-i})^u = g^i(u, \sigma^{-i}), u \in S^i$. Hence $g^i(\sigma) = \langle Vg^i(\sigma^{-i}), \sigma^i \rangle$ and $\sigma \in \Sigma$ is a Nash equilibrium iff:

$$\langle Vg^i(\sigma^{-i}), \sigma^i - \tau^i \rangle \geq 0, \qquad \forall \tau^i \in \Sigma^i, \qquad \forall i \in I.$$

Concave games

I is a finite set of players, for each $i \in I$, $X^i \subset H^i$(Hilbert) is the convex set of actions of player i and $G^i : S = \prod_j X^j \longrightarrow I\!R$ his payoff function.

Assume G^i concave and $\mathscr{C}^1$ w.r.t. x^i.

Then $x \in X$ is a Nash equilibrium iff:

$$\langle \nabla_i G^i(x), x^i - y^i \rangle_{H^i} \geq 0, \qquad \forall y^i \in X^i, \qquad \forall i \in I.$$

where ∇_i stands for the gradient of G^i w.r.t. x^i.

Population games

I is a finite set of populations of non atomic players; for each $i \in I$, S^i is the finite set of actions of population i and $X^i = \Delta(S^i)$ is the simplex over S^i. x^{iu} is the proportion of players in population i that play $u \in S^i$.

Given $K^i : X \longrightarrow I\!R^{S^i}$, $K^{iu}(x)$ is the payoff of a member of population i using the action $u \in S^i$ given the configuration x.

Then $x \in X$ is a Nash/Wardrop [48] equilibrium iff:

$$x^{iu} > 0 \Rightarrow K^{iu}(x) \geq K^{iv}(x), \quad \forall u, v \in S^i, \forall i \in I. \tag{7}$$

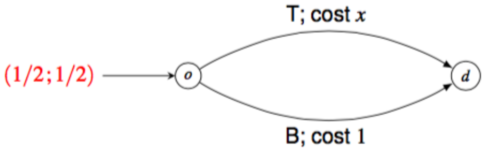
which is:

$$\langle K^i(x), x^i - y^i \rangle = \sum_{u \in S^i} K^{iu}(x)(x^{iu} - y^{iu}) \geq 0, \qquad \forall y^i \in X^i, \qquad \forall i \in I.$$

or

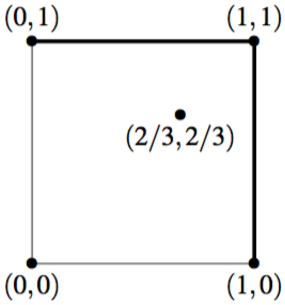
$$\langle K(x), x - y \rangle = \sum_i \langle K^i(x), x^i - y^i \rangle \geq 0, \qquad \forall y \in X$$

A typical example is congestion games: i corresponds to the type of the agent and u to the link in a network.
Consider the following Pigou's example.
Two roads, T and B link the origin o to the destination d. On T the cost is x if the congestion is x. On B there is a constant cost of 1.
Consider two populations of size $1/2$ each.



Since the agents are nonatomic they will all choose B is $x < 1$, hence the only equilibrium is $(0,0)$.
Note that in the case of two players controlling each a mass $1/2$ the only equlibrium is $(2/3,2/3)$.

Finally if the mass is not splittable, the players use mixed strategies and the set of equilibria is of the form: one player uses T and the other is indifferent between T and B.



Set of equilibria, as a function of the probability of T.

Finite composite games

The same approach can be extended to the case where different types of participants are present simultaneously (Sorin and Wang, 2016) [42].

General case

Consider a finite collection of convex compact sets $X^i \subset H^i$ (Hilbert), and evaluation mappings $\Phi^i : X \to H^i$, $i \in I$, with $X = \prod_j X^j$.

Definition

$NE(\Phi)$ is the set of $x \in X$ satisfying:

$$\langle \Phi(x), x - y \rangle \geq 0, \qquad \forall y \in X \tag{8}$$

where $\langle \Phi(x), x - y \rangle = \sum_i \langle \Phi^i(x), x^i - y^i \rangle_{H^i}$.

Remark that all the previous sets of equilibria can be written this way.

Let Π_X be the projection from H to the closed convex set X and $\mathbf{T}$ the map from X to itself defined by:

$$\mathbf{T}(x) = \Pi_X[x + \Phi(x)]$$

Proposition

$NE(\Phi)$ is the set of fixed points of $\mathbf{T}$.

Proof: The characterization of the projection gives:

$$\langle x + \Phi(x) - \Pi_X[x + \Phi(x)], y - \Pi_X[x + \Phi(x)]\rangle \leq 0, \qquad \forall y \in X,$$

hence $\Pi_C[x + \Phi(x)] = x$ is the solution iff $x \in NE(\Phi)$. ∎

Corollary

Assume Φ continuous on X. Then $NE(\Phi) \neq \emptyset$.

Proof: The map $x \mapsto \Pi_C[x + \Phi(x)]$ is continuous from the convex compact set X to itself, hence a fixed point exists. ∎

Supermodular games

Endow the euclidean space $I\!R^n$, with the product (partial) order $x \geq y$ iff $x_i \geq y_i$ for all i.
$S \subset I\!R^n$ is a *lattice* if for all $x, y \in S$: $\sup\{x, y\} \in S$ and $\inf\{x, y\} \in S$.
Recall the version of the fixed point theorem in this framework.

Theorem (Tarski, 1955 [43])

Let $S \subset I\!R^n$ be a non empty compact lattice and f an increasing function from S to itself.
Then f has a fixed point.

Consider a strategic game G, where for each $i \in I$, S^i is a non-empty compact subset of $I\!R^{m_i}$ and g^i is upper semi continuous in s^i for each fixed s^{-i}.
Assume moreover the game supermodular:
(i) For all i, S^i is a lattice in $I\!R^{m_i}$.
(ii) g^i has increasing differences in (s^i, s^{-i}):

$$g^i(s^i, s^{-i}) - g^i(s'^i, s^{-i}) \geq g^i(s^i, s'^{-i}) - g^i(s'^i, s'^{-i})$$

as soon as $s^i \geq s'^i$ and $s^{-i} \geq s'^{-i}$.
(iii) g^i is supermodular w.r.t. s^i: $\forall s^{-i} \in S^{-i}$,

$$g^i(s^i, s^{-i}) + g^i(s'^i, s^{-i}) \leq g^i(s^i \vee s'^i, s^{-i}) + g^i(s^i \wedge s'^i, s^{-i}).$$

Proposition (Topkis, 1979 [44])

Under the previous hypotheses, the game G has an equilibrium.

Proof: For each i and s^{-i}, $BR^i(s^{-i})$ is a non-empty compact lattice of $I\!R^{m_i}$. If $s^{-i} \geq s'^{-i}$, $\forall t'^i \in BR^i(s'^{-i})$, $\exists t^i \in BR^i(s^{-i})$ such that $t^i \geq t'^i$. Apply Tarski's theorem to the maximal element of the best reply map. ∎

Potential games

A) Finite case

A real function P defined on Σ is a potential (Monderer and Shapley, 1996 [25]) for the game (g, Σ) if:

$$g^i(s^i, u^{-i}) - g^i(t^i, u^{-i}) = P(s^i, u^{-i}) - P(t^i, u^{-i}), \forall s^i, t^i \in S^i, u^{-i} \in S^{-i}, \forall i \in I. \tag{9}$$

This means that the impact due to a change of action of player i is the same on g^i and on P, for all $i \in I$.

B) Evaluation functions

A real function W, of class $\mathscr{C}^1$ on a neighborhood Ω of X, is a potential for the game with evaluation Φ if for each $i \in I$, there is a strictly positive function $\mu^i(x)$ defined on X such that:

$$\left\langle \nabla_i W(x) - \mu^i(x)\Phi^i(x), y^i \right\rangle = 0, \quad \forall x \in X, \forall y^i \in TX^i, \forall i \in I, \tag{10}$$

where $TX^i = \{y \in I\!R^{S^i}, \sum_{p \in S^i} y_p = 0\}$ is the tangent space to X^i and ∇_i is the gradient w.r.t. x^i.

Theorem

Let $\Gamma(\Phi)$ be a game with potential W.
1. Every local maximum of W is an equilibrium of $\Gamma(\Phi)$.
2. If W is concave on X, then any equilibrium of $\Gamma(\Phi)$ is a global maximum of W on X.

Proof: The condition implied by a local maximum is

$$\langle \nabla W(x), x - y \rangle \geq 0, \quad \forall y \in X$$

hence in particular

$$\langle \nabla_i W(x), x^i - y^i \rangle \geq 0, \quad \forall y^i \in X^i$$

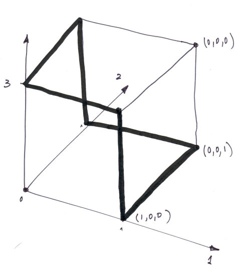
so that

$$\langle \nabla_i \Phi^i(x), x^i - y^i \rangle \geq 0, \quad \forall y^i \in X^i.$$

The inverse statement is clear. ∎

Minority game

Consider a three player game where each of the players has two strategies A, B. The payoff is one for the player being the only one to make his choice (if any) and 0 otherwise.
Pure equilibria are of the form (A, B, B) and mixed equilibria of the form $(A, B, ?)$ plus the completely symmetric one: $(1/2, 1/2)$ for each player. The set of equilibria is homeomorphic to a circle plus an isolated point.



This is a potential game.

Dissipative games

The game with evaluation Φ is dissipative if Φ satisfies:

$$\langle \Phi(x) - \Phi(y), x - y \rangle \leq 0, \qquad \forall\, (x,y) \in X \times X.$$

Hofbauer and Sandholm [23] use the terminology "stable games".
Let $SNE(\Phi)$ be the set of $x \in X$ satisfying:

$$\langle \Phi(y), x - y \rangle \geq 0, \qquad \forall y \in X.$$

Proposition

If $\Gamma(\Phi)$ is dissipative and Φ is continuous:

$$SNE(\Phi) = NE(\Phi).$$

in particular $NE(\Phi)$ is convex.

Proof: One direction is clear and does not use continuity. If Φ is dissipative and x is an equilibrium, then:

$$\langle \Phi(y), x-y \rangle \geq \langle \Phi(x), x-y \rangle \geq 0, \quad \forall y \in X.$$

On the other hand, given $z \in X$ and $t \in (0,1]$, let $y = x + t(z-x)$, hence:

$$\langle \Phi(x+t(z-x)), t(x-z) \rangle \geq 0.$$

Dividing by t and then letting t go to 0 gives, by continuity of Φ, the result. ∎

Correlated equilibria

This section deals with “correlated equilibrium” (Aumann, 1974 [1]) which is an extension of Nash equilibrium that has good properties from strategic, analytic and dynamic viewpoints.

Example 1

$3,1$	$0,0$
$0,0$	$1,3$

This game has two 2 pure equilibria that are not symmetrical and a mixed, symmetric $(1/4, 3/4)$ and dominated in terms of payoff.

The use a public fair coin and of the following plan: $(3,1)$ if Head and $(1,3)$ if Tail, induces the following distributions on profiles:

$1/2$	0
0	$1/2$

Example 2

	l	r
T	$2,7$	$6,6$
B	$0,0$	$7,2$

Introduce a signal space $\Omega = (X, Y, Z)$, with uniform probability $(1/3, 1/3, 1/3)$. Assume that the players get private messages: 1 knows $a = \{X, Y\}$ or $b = \{Z\}$, 2 knows $\alpha = \{X\}$ or $\beta = \{Y, Z\}$. Consider the strategies: T if a, B if b for player 1; l if α, r if β for player 2.

They induce on the set of profiles S the correlation matrix:

$1/3$	$1/3$
0	$1/3$

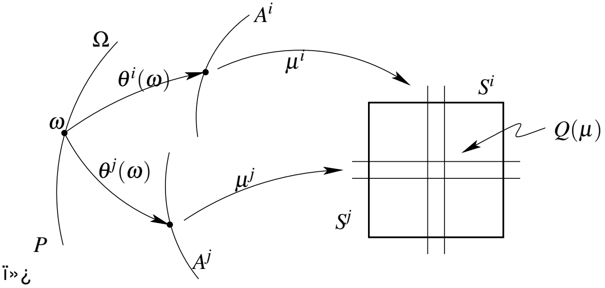
and no deviation is profitable.

The formal model is as follows.
An information structure $\mathscr{I}$ is given by:
- a probability space $(\Omega, \mathscr{A}, P)$
- a measurable map θ^i from $(\Omega, \mathscr{A})$ to A^i (signals of i), for each $i \in I$.

Let G be a finite game defined by $g : S = \Pi_i S^i \to I\!R^n$.
The game G extended by $\mathscr{I}$, denoted $[G, \mathscr{I}]$, is the game played in 2 stages:
- stage 0 : the random variable ω is selected according to P and the signal $\theta^i(\omega)$ is sent to player i.
- stage 1 : the players play in G.

A strategy μ^i of player i in the game $[G, \mathscr{I}]$ is a map from A^i to S^i.
A profile μ of such elements is called a correlated strategy.

A correlated equilibrium of G is a Nash equilibrium of some extended game $[G, \mathscr{I}]$.
A profile μ of strategies in $[G, \mathscr{I}]$ maps the probability P on Ω to a probability $Q(\mu)$ on S: random variable $\to$ signals $\to$ profile of moves.



Explicitly, for each ω, $Q(\omega, \mu)$ is the probability on S given by $\prod_i \mu^i(\theta^i(\omega))$ and $Q(\mu)$ is the expectation w.r.t. P.

$CED(G)$ is the set of correlated equilibria distributions in G:

$$CED(G) = \cup_{\mathscr{I},\mu}\{Q(\mu); \mu \text{ equilibrium in } [G,\mathscr{I}]\}$$

$CED(G)$ is a convex set: simply consider the convex combination of information structures.

A canonical information structure for G is given by:
$\Omega = S$; $\theta^i : S \to S^i, \theta^i(s) = s^i$. P is a probability on S and each player is informed upon his component.
A canonical correlated equilibrium is an equilibrium of $[G,\mathscr{I}]$ where $\mathscr{I}$ is a canonical information structure and where equilibrium strategies are given by:

$$\mu^i(\omega) = \mu^i(s) = \mu^i(s^i) = s^i.$$

“Each player follows his signal".
The induced canonical correlated equilibrium distribution ($CCED$) is obviously P.

Theorem (Aumann, 1974 [1])

$$CCED(G) = CED(G)$$

Proof : Let μ be an equilibrium profile in an extension $[G, \mathscr{I}]$ and $Q = Q(\mu)$ the induced distribution.
Then Q belongs to $CCED(G)$.
In fact, each player i get less information: his move s^i rather than the signal a^i such that $\mu^i(a^i) = s^i$. But s^i is a best reply to the correlated strategy of $-i$, conditional to a^i. It is then enough to use the convexity of BR^i on $\Delta(S^{-i})$. ■

The characterization in the finite case is given by:

Theorem

$CED = \cap_i CED^i$ *with* $Q \in CED^i$ *iff:*

$$\forall s^i, t^i \in S^i \qquad \sum_{s^{-i} \in S^{-i}} [g^i(s^i, s^{-i}) - g^i(t^i, s^{-i})] Q(s^i, s^{-i}) \geq 0.$$

Proof : Let $Q \in CCED(G)$.
Assume s^i is announced (i.e. its marginal $Q^i(s^i) = \sum_{s^{-i}} Q(s^i, s^{-i}) > 0$) and consider the conditional distribution on S^{-i}, $Q(.|s^i)$, then the equilibrium condition writes:

$$s^i \in BR^i(Q(.|s^i).$$

s^i is a best reply of player i to the distribution of the moves of $-i$, conditional to s^i. ∎

The approach in terms of Nash equilibrium of an extended game is "ex-ante".
The previous characterization corresponds to an "ex-post" criteria.

Corollary

The set of CED is the convex hull of finitely many points.

Proof : It is a subset of $\Delta(S)$ defined by a finite set of linear inequalities. ∎

Comments

1) There exists correlated equilibrium distributions outside the convex hull of Nash equilibria. In the game:

$0,0$	$5,4$	$4,5$
$4,5$	$0,0$	$5,4$
$5,4$	$4,5$	$0,0$

the only equilibrium is symmetric: $(1/3,1/3,1/3)$ with payoff 3. The following is a CED:

0	$1/6$	$1/6$
$1/6$	0	$1/6$
$1/6$	$1/6$	0

inducing the payoff 9/2.

2) We provide an elementary proof of existence of correlated equilibria via the minmax theorem: CED corresponds to the set of optimal strategies of a player in a finite 0-sum game, Hart and Schmeidler (1989) [20].
We make the computations in the two player case. The extension to more players is straightforward.

Let G be a strategic 2 player game with strategy sets S^1 and S^2 and payoff $g : S = S^1 \times S^2 \longrightarrow I\!R^2$. Consider now the game Γ which is 2 player, zero-sum with strategy sets S and $L = (S^1)^2 \cup (S^2)^2$ and payoff γ defined by:

$$\gamma(s; t^i, u^i) = (g^i(t^i, s^{-i}) - g^i(u^i, s^{-i}))\mathbf{1}_{\{t^i = s^i\}}$$

Γ has a value v and optimal strategies.
If $v \geq 0$ and $Q \in \Delta(S)$ is an optimal strategy of player 1, then Q is a correlated equilibrium distribution in G.

Let $\pi \in \Delta(L)$. Define ρ^1, a transition probability on S^1, by:

$$\rho^1(t^1;u^1) = \pi(t^1,u^1), \quad \text{if } t^1 \neq u^1$$

$$\rho^1(t^1;t^1) = 1 - \sum_{u^1 \neq t^1} \pi(t^1,u^1).$$

Let now μ^1 be a probability on S^1 invariant by ρ^1:

$$\mu^1(t^1) = \sum_{u^1} \mu^1(u^1)\rho(u^1;t^1).$$

Define ρ^2 and μ^2 similarly and let $\mu = \mu^1 \times \mu^2$.
Note that the payoff $\gamma(\mu;\pi)$ can be decomposed as follows:

$$\gamma(\mu,\pi) = \sum_{(s^1,s^2)\in S^1\times S^2} \mu^1(s^1)\mu^2(s^2) \sum_{i=1,2} \sum_{(t^i,u^i)\in L^i} \pi(t^i,u^i)\gamma((s^1,s^2);t^i,u^i).$$

Let A_1 be the term corresponding to $i = 1$.

$$
\begin{aligned}
A_1 &= \sum_{s^1,s^2} \mu^1(s^1)\mu^2(s^2) \sum_{(t^1,u^1)} \pi(t^1,u^1)\gamma((s^1,s^2);t^1,u^1) \\
&= \sum_{s^1} \mu^1(s^1) \sum_{u^1} \pi(s^1,u^1) \sum_{s^2} \mu^2(s^2)\gamma((s^1,s^2);s^1,u^1) \\
&= \sum_{s^1} \mu^1(s^1) \sum_{u^1 \neq s^1} \rho^1(s^1,u^1) \sum_{s^2} \mu^2(s^2)(g^1(s^1,s^2) - g^1(u^1,s^2)) \\
&= \sum_{s^1} \mu^1(s^1) \sum_{u^1} \rho^1(s^1,u^1)(g^1(s^1,\mu^2) - g^1(u^1,\mu^2)) \\
&= \sum_{s^1,u^1} \mu^1(s^1)\rho^1(s^1,u^1)g^1(s^1,\mu^2) - \sum_{s^1,u^1} \mu^1(s^1)\rho^1(s^1,u^1)g^1(u^1,\mu^2) \\
&= \sum_{s^1} \mu^1(s^1)g^1(s^1,\mu^2) - \sum_{u_1} \mu^1(u^1)g^1(u^1,\mu^2) \\
&= g^1(\mu^1,\mu^2) - g^1(\mu^1,\mu^2) = 0
\end{aligned}
$$

Similarly $A_2 = 0$, hence $\gamma(\mu,\pi) = 0$.
$\forall \pi \in \Delta(L), \exists \mu \in \Delta(S)$ such that $\gamma(\mu,\pi) \geq 0$, hence the value of Γ is non negative, hence the result. ■

3) A superset of correlated equilibria which is important in applications is the Hannan set.
Like the set of CED it is a subset of $\Delta(S)$ obtained by intersection : $H = \cap_i H^i$ with:

$$H^i = \{Q \in \Delta(S); g^i(s^i, Q^{-i}) \leq g^i(Q), \quad \forall s^i \in S^i\}$$

This corresponds to be imune to deviation from Q before getting the signal.
In the case of a zero-sum game one has, for $z \in H$ with marginals z^1, z^2:

$$f(z) \geq f(s^1, z^2), \quad \forall s^1 \in S^1$$

and the opposite inequality for the other player hence the marginals z^1, z^2 are optimal strategies and $f(z)$ is equal to the value.

Example: for the game:

0	1	-1
-1	0	1
1	-1	0

the distribution:

$1/3$	0	0
0	$1/3$	0
0	0	$1/3$

is in the Hannan set.

Stochastic approximation for differential inclusions

We summarize here results from Benaïm, Hofbauer and Sorin (2005), following the approach for ODE by Benaïm (1996, 1999), Benaïm and Hirsch (1996).

1. Differential inclusions
Given a correspondence F from $\mathbb{R}^m$ to itself, consider the differential inclusion

$$\dot{\mathbf{x}} \in F(\mathbf{x}). \qquad (I)$$

It induces a set-valued dynamical system $\{\Phi_t\}_{t \in \mathbb{R}}$ defined by

$$\Phi_t(x) = \{\mathbf{x}(t) : \mathbf{x} \text{ is a solution to } (I) \text{ with } \mathbf{x}(0) = x\}.$$

We also write $\mathbf{x}(t) = \varphi_t(x)$ and define $\Phi_A(B) = \cup_{t \in A,\, x \in B} \Phi_t(x)$.

2. Attractors

Definition

1) C is invariant if for any $x \in C$ there exists a complete solution: $\varphi_t(x) \in C$ for all $t \in I\!R$.
2) C is attracting if it is compact and there exist a neighborhood U, $\varepsilon_0 > 0$ and a map $T : (0, \varepsilon_0) \to I\!R^+$ such that: for any $y \in U$, any solution φ, $\varphi_t(y) \in C^\varepsilon$ for all $t \geq T(\varepsilon)$, i.e.

$$\Phi_{[T(\varepsilon),+\infty)}(U) \subset C^\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

U is a uniform basin of attraction of C and we write $(C;U)$ for the couple.
3) C is an attractor if it is attracting and invariant.
4) The ω-limit set of C is defined by

$$\omega_\Phi(C) = \cap_{s\geq 0}\overline{\cup_{y\in C}\cup_{t\geq s}\Phi_t(y)} = \cap_{s\geq 0}\overline{\Phi_{[s,+\infty)}(C)}. \qquad (11)$$

5) Given a closed invariant set L, the induced set-valued dynamical system is denoted by Φ^L. L is attractor free if Φ^L has no proper attractor.

3. Lyapounov functions

We describe here practical criteria for attractors.

Proposition

Let A be a compact set, U be a relatively compact neighborhood and V a function from $\overline{U}$ to $I\!R^+$. Assume:
i) $\Phi_t(U) \subset U$ for all $t \geq 0$.
ii) $V^{-1}(0) = A$
iii) V is continuous and strictly decreasing on trajectories on $\overline{U} \setminus A$:

$$V(x) > V(y), \quad \forall x \in U \setminus A, \forall y \in \Phi_t(x), \quad \forall t > 0.$$

Then:
a) A is Lyapounov stable and $(A;U)$ is attracting.
b) $(B;U)$ is an attractor for some $B \subset A$.

Definition

A real continuous function V on U open in $\mathbb{R}^m$ is a Lyapunov function for (A,U), $A \subset U$ if :
$V(y) < V(x)$ *for all* $x \in U \setminus A, y \in \Phi_t(x), t > 0,$
$V(y) \leq V(x)$ *for all* $x \in A, y \in \Phi_t(x)$ *and* $t \geq 0.$

Proposition

Suppose V is a Lyapunov function for (A,U). Assume that $V(A)$ has empty interior. Let L be a non empty, compact, invariant and attractor free subset of U. Then L is contained in A and $V_{|L}$ is constant.

3. Asymptotic pseudo-trajectories

Definition

A continuous function $\mathbf{z} : I\!R^{+} \to I\!R^{m}$ is an asymptotic pseudo-trajectory (APT) for (I) if for all T

$$\lim_{t\to\infty} \inf_{\mathbf{x}\in S_{\mathbf{z}(t)}} \sup_{0\le s\le T} \|\mathbf{z}(t+s)-\mathbf{x}(s)\| = 0. \tag{12}$$

where S_x denotes the set of solutions of (I) starting from x at 0.

In other words, for each fixed T, the curve: $s \to \mathbf{z}(t+s)$ from $[0,T]$ to $I\!R^{m}$ shadows some trajectory for (I) of the point $\mathbf{z}(t)$ over the interval $[0,T]$ with arbitrary accuracy, for sufficiently large t.

Let

$$L(\mathbf{z}) = \bigcap_{t\ge 0} \overline{\{\mathbf{z}(s) : s \ge t\}}$$

be the limit set.

Theorem

Let $\mathbf{z}$ be a bounded APT of (I). Then $L(\mathbf{z})$ is (internally chain transitive, hence) compact, invariant and attractor free.

4. Perturbed solutions

Definition

A continuous function $\mathbf{y} : I\!R_+ = [0,\infty) \to I\!R^m$ *is a perturbed solution to* (I) *if it satisfies the following set of conditions* (II)*:*

i) $\mathbf{y}$ *is absolutely continuous.*

ii) There exists a locally integrable function $t \mapsto U(t)$ *such that* $\lim_{t\to\infty} \sup_{0\le v\le T} \left\| \int_t^{t+v} U(s)\, ds \right\| = 0$*, for all* $T > 0$*.*

iii)

$$\dot{\mathbf{y}}(t) \in F^{\delta(t)}(\mathbf{y}(t)) + U(t),$$

for almost every $t > 0$*, for some function* $\delta : [0,\infty) \to I\!R$ *with* $\delta(t) \to 0$ *as* $t \to \infty$*.*

Here $F^{\delta}(x) := \{y \in I\!R^m : \exists z : \|z - x\| < \delta,\ d(y, F(z)) < \delta\}$.

The purpose is to investigate the long-term behavior of $\mathbf{y}$ and to describe its limit set $L(\mathbf{y})$ in terms of the dynamics induced by F.

Theorem

Any bounded solution $\mathbf{y}$ *of* (II) *is an APT of* (I)*.*

A natural class of perturbed solutions to F arises from certain stochastic approximation processes.

Definition

A discrete time process $\{x_n\}$ *with values in* $I\!R^m$ *is a* (γ, U) *discrete stochastic approximation for* (I) *if it verifies a recursion of the form*

$$x_{n+1} - x_n \in \gamma_{n+1}[F(x_n) + U_{n+1}], \qquad (III)$$

where the characteristics $\{\gamma_n\}$ *and* $\{U_n\}$ *satisfy*
i) $\{\gamma_n\}_{n\geq 1}$ *is a sequence of nonnegative numbers such that*

$$\sum_n \gamma_n = \infty, \qquad \lim_{n\to\infty} \gamma_n = 0;$$

ii) $U_n \in I\!R^m$ *are (deterministic or random) perturbations.*

To such a process is naturally associated a continuous time interpolated (random) process $\mathbf{w}$ as usual (IV).

5. From interpolated process to perturbed solutions
The next result gives sufficient conditions on the characteristics of the discrete process (III) for its interpolation (IV) to be a perturbed solution (II).

Proposition

Assume that :
$(*)$ *For all* $T > 0$

$$\lim_{n\to\infty} \sup \left\{ \left\| \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\| : k = n+1, \ldots, m(\tau_n + T) \right\} = 0,$$

where $\tau_n = \sum_{i=1}^{n} \gamma_i$ *and* $m(t) = \sup\{k \geq 0 : t \geq \tau_k\}$;
$(**)$ $\sup_n \|x_n\| = M < \infty$.
Then the interpolated process $\mathbf{w}$ *is a perturbed solution of* (I).

We describe now sufficient conditions for condition $(*)$ to hold. Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\mathscr{F}_n\}_{n\geq 0}$ a filtration of $\mathscr{F}$. A stochastic process $\{x_n\}$ satisfies the Robbins–Monro condition if:

i) $\{\gamma_n\}$ is a deterministic sequence.

ii) $\{U_n\}$ is *adapted* to $\{\mathscr{F}_n\}$,

iii) $\mathbf{E}(U_{n+1} \mid \mathscr{F}_n) = 0$.

Proposition

Let $\{x_n\}$ given by (III) be a Robbins–Monro process. Suppose that for some $q \geq 2$

$$\sup_n \mathbf{E}(\|U_n\|^q) < \infty \quad \textit{and} \quad \sum_n \gamma_n^{1+q/2} < \infty.$$

Then assumption $(*)$ holds with probability 1.

Remark Typical applications are

i) U_n uniformly bounded in L^2 and $\gamma_n = \frac{1}{n}$,

ii) U_n uniformly bounded and $\gamma_n = o(\frac{1}{\log n})$.

6. Main result

Consider a random discrete process defined on a compact subset of $I\!R^K$ and satisfying the differential inclusion :

$$Y_n - Y_{n-1} \in a_n[T(Y_{n-1}) + W_n]$$

where

i) T is an u.s.c. correspondence with compact convex values

ii) $a_n \geq 0$, $\sum_n a_n = +\infty$, $\sum_n a_n^2 < +\infty$

iii) $E(W_n | Y_1, ..., Y_{n-1}) = 0$.

Theorem

The set of accumulation points of $\{Y_n\}$ is almost surely a compact set, invariant and attractor free for the dynamical system defined by the differential inclusion:

$$\dot{Y} \in T(Y).$$

A typical application is the case where:

$$Y_n - Y_{n-1} \in a_n \mathbf{T}(Y_{n-1})$$

with $\mathbf{T}$ random, where one writes

$$Y_n - Y_{n-1} \in a_n [E[\mathbf{T}(Y_{n-1}) | Y_1, ..., Y_{n-1}]$$

$$+(\mathbf{T}(Y_{n-1}) - E[\mathbf{T}(Y_{n-1}) | Y_1, ..., Y_{n-1}])]$$

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